(I)

Why $\delta(x)\cdot f(x) = f(0)\cdot\delta(x)$?

Note

1.) $f(x)$ can be any function continuous at where the delta function gives a spike. (condition)

2.) $f(x)$ need not to be a test function.

$\delta(x)\cdot f(x)$: ∞ · $f(0)$

$$\delta(x)\cdot f(x) = \begin{cases} 0 & , x\neq 0 \\ \pm\infty & , x=0 \end{cases}$$

area $= f(0)$

$f(0)\cdot\delta(x)$ → same

$$\int_{-\infty}^{\infty}\delta(x)\cdot f(x)\,dx = \int_{0^-}^{0^+}\delta(x)\cdot f(x)\,dx = f(0)\cdot\int_{0^-}^{0^+}\delta(x)\,dx = f(0)$$

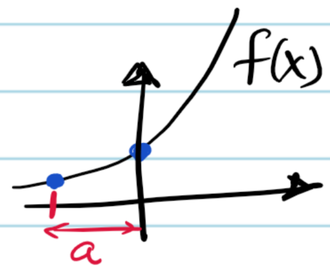
$(0^-, 0^+)$ $\quad f(0^-) = f(0^+) = f(0)$

Graphs

same area but shifted

① $\delta(x)\cdot f(x) = f(0)\cdot\delta(x)$

② $\delta(x-a)\cdot f(x-a) = f(0)\cdot\delta(x-a)$

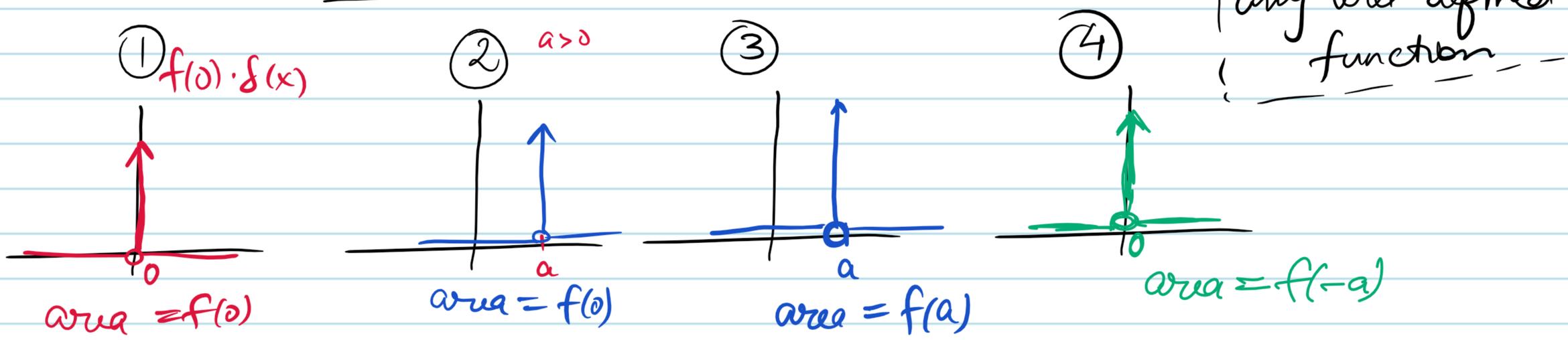


③ $\delta(x-a)\cdot f(x) = f(a)\cdot\delta(x-a)$

④ $\delta(x)\cdot f(x-a) = f(-a)\cdot\delta(x)$

same area but shifted

where f can be any well-defined function



II Scaling Property

$$\delta(\alpha x) = \frac{\delta(x)}{|\alpha|}, \quad \alpha \neq 0.$$

$\alpha = -1 \Rightarrow \delta(-x) = \delta(x)$ (even symmetry)

$\delta(\alpha x)$, $\alpha \neq 0$

$$\langle \delta(\alpha x), \phi(x) \rangle = \int_{-\infty}^{\infty} \delta(\alpha x)\, \phi(x)\, dx$$

$\alpha x = u$, $dx = \frac{du}{\alpha}$

$\alpha > 0$: $\alpha = |\alpha|$, $(-\infty, \infty)$

$$= \int_{-\infty}^{\infty} \delta(u)\, \phi\left(\frac{u}{|\alpha|}\right) \frac{du}{|\alpha|}$$

$\alpha > 0$, $\delta(\alpha x) = \frac{\delta(x)}{|\alpha|}$

$$= \frac{1}{|\alpha|} \left\langle \delta(u), \phi\left(\frac{u}{|\alpha|}\right) \right\rangle$$

$$= \frac{\phi(0)}{|\alpha|} = \left\langle \frac{\delta(x)}{|\alpha|}, \phi(x) \right\rangle$$

$\alpha < 0$: $\alpha = -|\alpha|$, $(\infty, -\infty)$

$$+ \int_{-\infty}^{\infty} \delta(u)\, \phi\left(\frac{u}{-|\alpha|}\right) \cdot \frac{du}{+|\alpha|}$$

$$= \frac{1}{|\alpha|} \langle \delta(u), \phi\left(\frac{u}{-|\alpha|}\right) \rangle$$

$$= \left\langle \frac{\delta(x)}{|\alpha|}, \phi(x) \right\rangle$$

III Shifting property

{ of $\delta_a(x)$ on a function $f(x)$ }

$$\langle \delta(x-a), \phi(x) \rangle = \phi(a)$$

$$\delta(x-a) = \delta_a(x)$$

$$\langle \delta_a(x), \phi(x) \rangle = \phi(a)$$

Translation/Shifting property :

$$\int_{-\infty}^{\infty} \delta(x-a) \cdot f(x)\, dx = f(a)$$

(provided f is continuous at $x=a$)

Q. $\langle \delta(ax+b), \phi(x) \rangle = ?$ given $a \neq 0$.

$$\delta(ax+b) = \delta\left(a\left(x + \frac{b}{a}\right)\right) = \frac{\delta(x+b/a)}{|a|}$$

$$= \frac{1}{|a|} \langle \delta(x+b/a), \phi(x) \rangle = \frac{\phi(-b/a)}{|a|}$$

Convolution

$$f(x) * g(x) \overset{def}{=} \int_{-\infty}^{\infty} f(\tau)\, g(x-\tau)\, d\tau = g(x) * f(x)$$

$= h(x)$

IV

Application of Shifting property of δ:

Convolution of $f(t)$ with $\delta_T(t)$: $= \delta(t-T)$

$$f(t) * \delta_T(t) = f(t) * \delta(t-T)$$

$$= \int_{-\infty}^{\infty} f(\tau) \cdot \delta_T(t-\tau)\, d\tau$$

$t-\tau-T=0 \Rightarrow \tau = t-T$

$$= \int_{-\infty}^{\infty} f(\tau) \cdot \delta((t-\tau-T))\, d\tau$$

$$= f(\tau)\Big|_{\tau=t-T} = f(t-T)$$

$\therefore$, $f(t) * \delta(t-T) = f(t-T)$

Identity:

$$\int_{-\infty}^{\infty} \delta(\alpha_1 - x)\, \delta(x-\alpha_2)\, dx = \delta(\alpha_1-\alpha_2)$$

(δ: $\delta(\alpha_1 - x)$; f: $\delta(x-\alpha_2)$; $x=\alpha_1$)

Shifting property of $\delta_a(x)$ on a distribution $T(x)$

Is applicable only when $T(x)$ is a tempered distribution

("δ" is a tempered distribution)

$$\int_{-\infty}^{\infty} \delta(x-a)\, f(x)\, dx = f(a)$$

f is a real continuous function at $x=a$.

(V)

$$x \cdot \delta(x) = 0_F$$

$$\Downarrow$$

$$x^n \cdot \delta(x) = 0_F$$

$$\Downarrow$$

$$(x-a)^n \cdot \delta(x-a) = 0_F$$

because

$$\langle x\delta(x), \phi(x) \rangle = \langle \delta(x), \phi(x) x \rangle = \langle 0, \phi(x) \rangle = 0_F$$

Note $\quad xT(x) = 0$

The soln of $x \cdot T(x) = 0_F$

is $T(x) = c \cdot \delta(x)$ only.

(for some $c \neq 0$)

$$x\,T(x) = x\,c\,\delta(x) = c \cdot x\delta(x) = 0$$

but $\quad x^n \cdot \delta(x-a) = ?$

$$\langle x^n \delta(x-a), \phi(x) \rangle = \langle \delta(x-a), x^n \phi(x) \rangle = a^n \cdot \phi(a) \neq 0$$

If $\quad x\,T_1(x) = x\,T_2(x)$, then $\quad T_1(x) = T_2(x) + c \cdot \delta(x)$.

(given) (to show)

$$x\,T(x) = 0_F \implies T(x) = c \cdot \delta(x) \quad (c \neq 0)$$

$$x \cdot \overbrace{(T_2(x) - T_1(x))} = 0$$

$$x \cdot T_3(x) = 0_F \implies T_3(x) = c \cdot \delta(x), \quad c \neq 0$$

$$T_2(x) - T_1(x) = c \cdot \delta(x), \quad c \neq 0$$

$$T_2(x) = T_1(x) + c \cdot \delta(x), \quad c \neq 0$$

$$T_1(x) = T_2(x) + c_2 \cdot \delta(x) \qquad c_2 \neq 0$$

1. Graphs $\delta(x)\cdot f(x)$ $\delta(x-a)\cdot f(x)$ $\delta(x-a)\cdot f(x-a)$ $\delta(x)\cdot f(x-a)$

2. Scaling $\rightarrow \delta(\alpha x) = \frac{\delta(x)}{|\alpha|}$, $\alpha \neq 0$ ←

3. Shifting $\rightarrow \delta(x-a)$ or $\delta_a(x)$ ←

4. Application (Scaling & Shifting), Convolution $f * \delta_a(x)$.

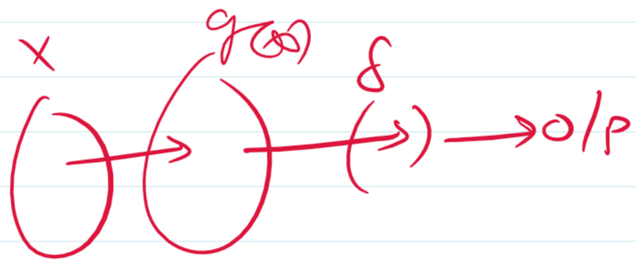
5. $x^n \cdot \delta(x) = 0_F$, $x \cdot T(x) = 0_F \Rightarrow T(x) = c\cdot\delta(x)$

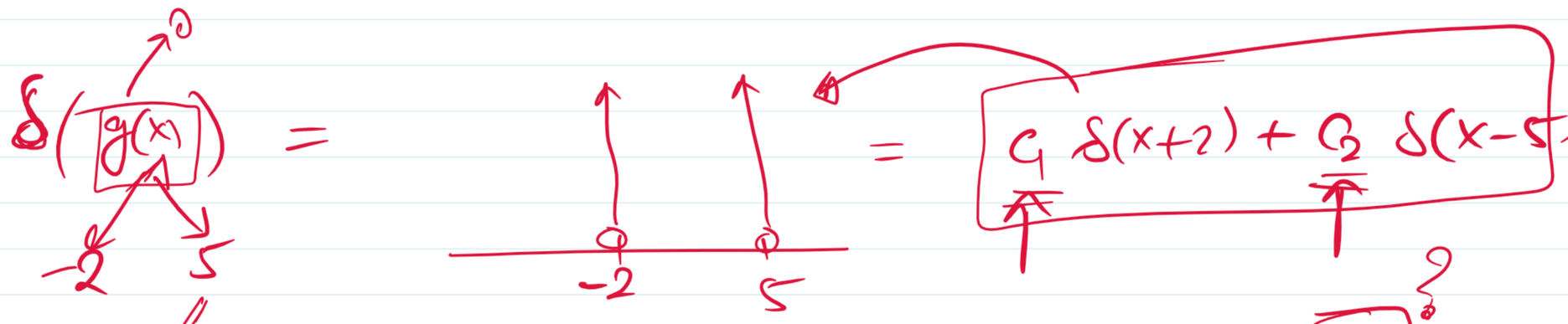
$\Rightarrow \boxed{\delta(x^2-4) = \frac{1}{4}\left(\delta(x-2)+\delta(x+2)\right)}$

question → formula : $\delta(x^2-\alpha^2) = \frac{1}{2\cdot|\alpha|}\cdot\left(\delta(x-\alpha)+\delta(x+\alpha)\right)$

$x, \alpha \in \mathbb{R}$

property - 6 Composition $\delta(g(x))$

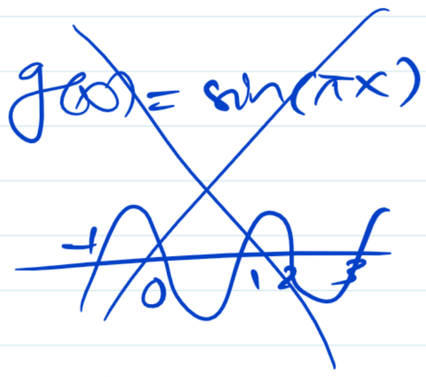




$\delta(g(x)) = \sum_i c_i \delta(x-x_i)$

$\delta(k(x+2)(x-5))$

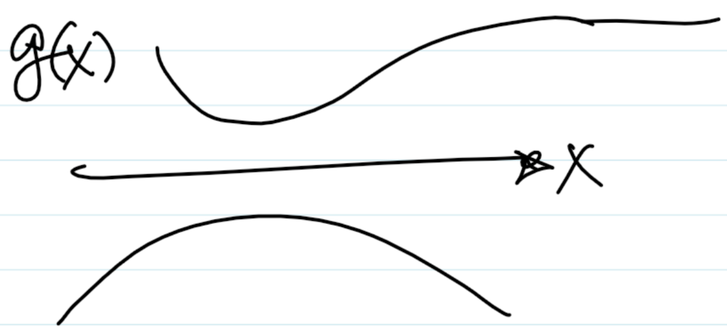
$g(x): \mathbb{R} \rightarrow \mathbb{R}$



[$\delta(g(x))$ is complicated. We will apply certain conditions on $g(x)$ near its roots to simplify $\langle \delta(g(x)), \phi(x) \rangle$.

$\delta(g(x))$

- $g(x)$ = has NO real roots → $\delta(g(x)) = 0_F$
- (I) $g(x)$ has finite no. of real roots
  - (II) non-repeated (simple) roots of $g(x)=0$
  - repeated



$$\int_{\mathbb{R}} = \int_{-\infty}^{\infty}$$

$$\int_{\text{Set } S} = \int_{S_{min}}^{S_{max}}$$

$S_{max} \to 5$

$S_{min} \to 2$

$[2,5)$

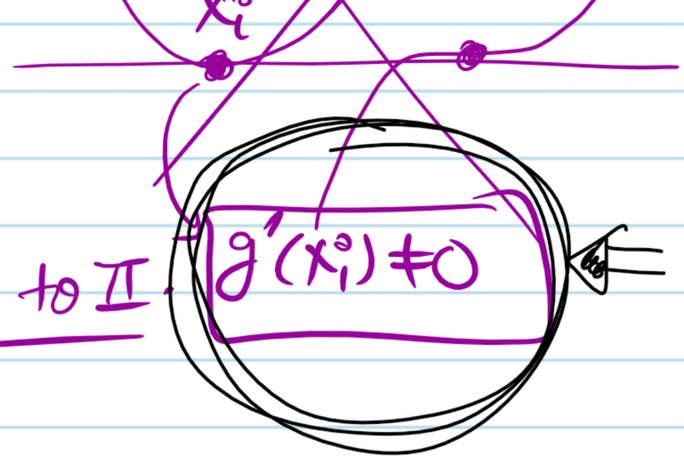
Assumptions

$g(x)=0$

(I) → finite nos of real roots

(II) → simple roots (non-repeated roots)

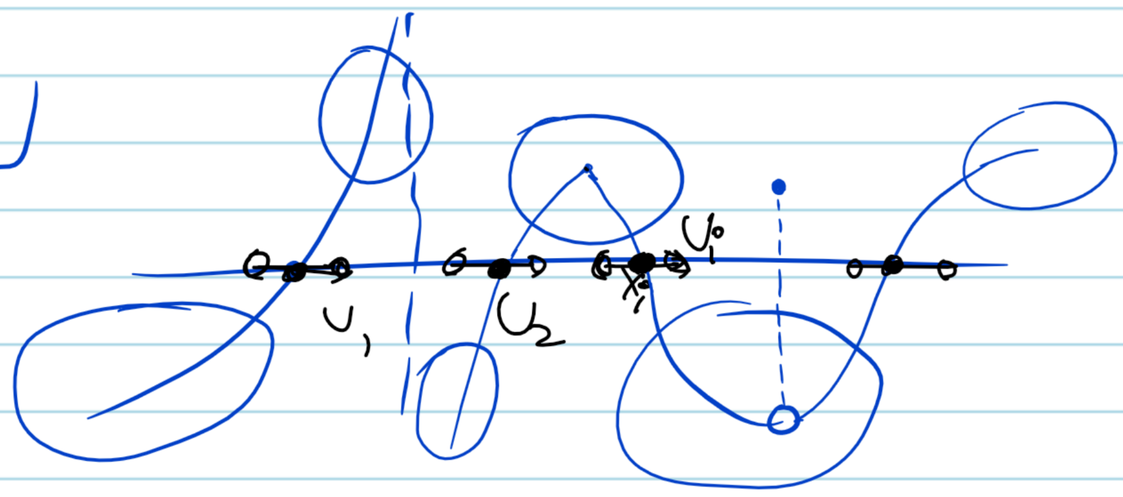
(III) → near all roots x_i, $x \in U_i$, the derivative $g'(x)$ exists.



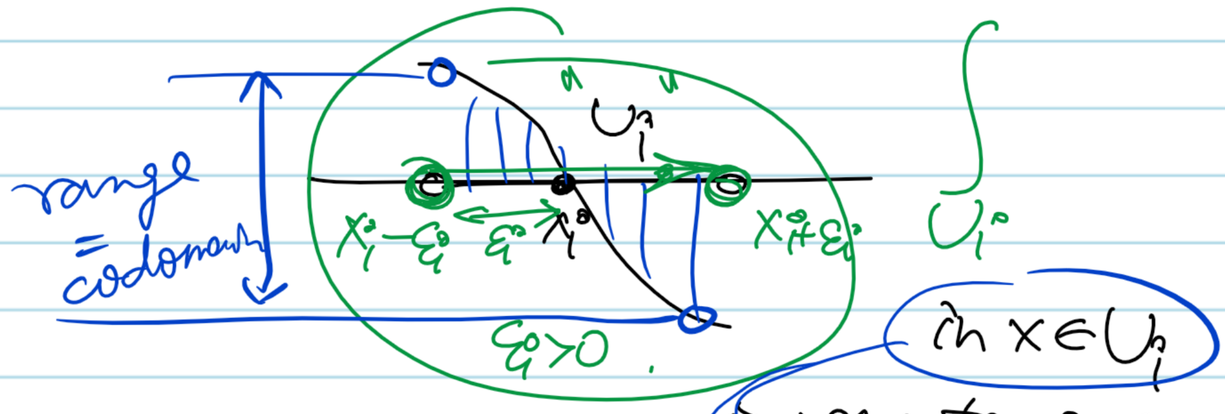
$$\langle \delta(g(x)), \phi(x) \rangle := \int_{\mathbb{R}} \delta(g(x)) \phi(x) dx$$

$u = g(x)$

$x = g^{-1}(u)$



$$= \sum_i \int_{U_i} \delta(g(x)) \phi(x) dx$$



in $x \in U_i$:

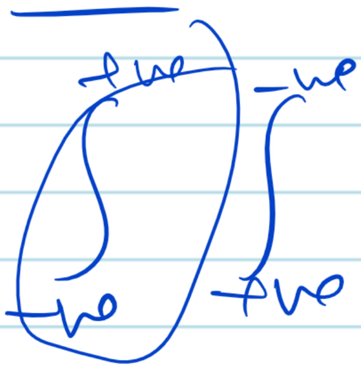
1) $g(x)$ is locally strictly monotonous.

2) $g^{-1}(x)$ exists.

in $x \in U_i$:

one-to-one

onto



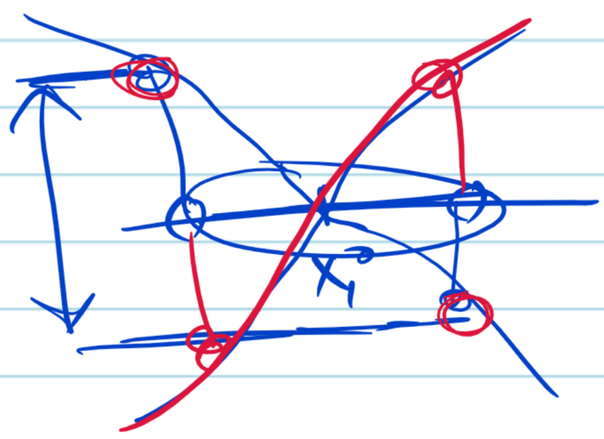
$$\int_{U_i} \delta(g(x)) \phi(x) dx = \int_{g(x_i - \varepsilon_i)}^{g(x_i + \varepsilon_i)} \delta(u) \frac{\phi(g^{-1}(u))}{g'(g^{-1}(u))} du$$

$g(x) = u$

$U_i = x_i + \varepsilon_i$

$LL = x_i - \varepsilon_i$

$$dx = \frac{du}{g'(x)} = \frac{du}{g'(g^{-1}(u))}$$

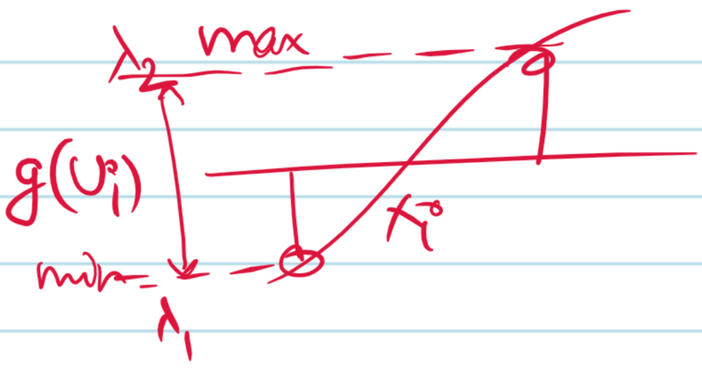


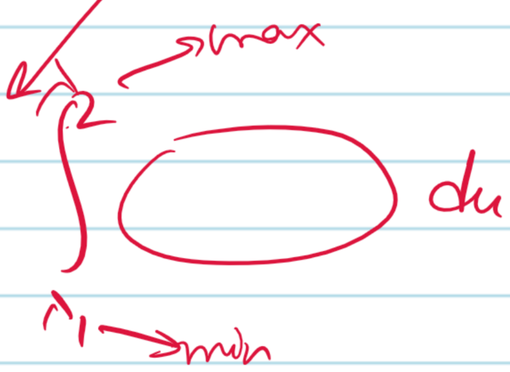
$g(x_i^0+\varepsilon_i^0)$

$g(x_i^0-\varepsilon_i^0)$

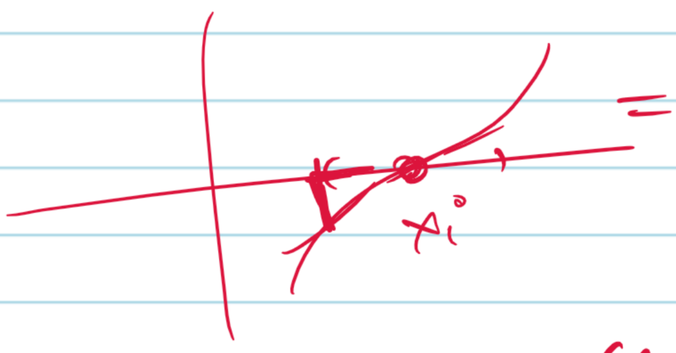
$$\int \delta(u)\,\frac{\phi(\bar g^{-1}(u))}{g'(\bar g^{-1}(u))}\,du$$

g is ↑ strictly in U_i^0



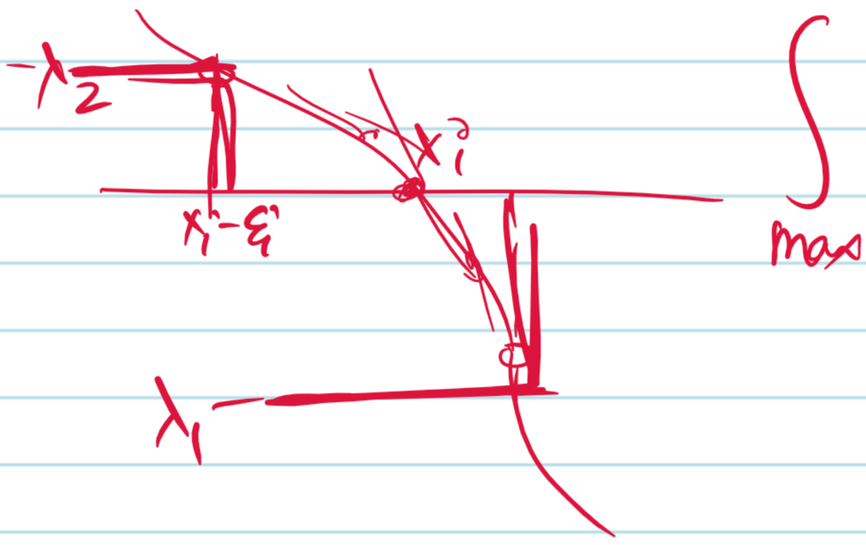


$$= \int_{g(U_i^0)} du$$



$$= \underline{g(U_i^0)}$$

g is ↓ strictly in U_i^0



max ← $g(x_i-\varepsilon_i)$

$$= -\int_{g(x_i+\varepsilon_i)} \delta(u)\,\frac{\phi(\bar g^{-1}(u))\,du}{g'(\bar g^{-1}(u))}$$

min ← $g(x_i+\varepsilon_i)$

$g'(\bar g^{-1}(u))$ → −ve

(↓ strictly)

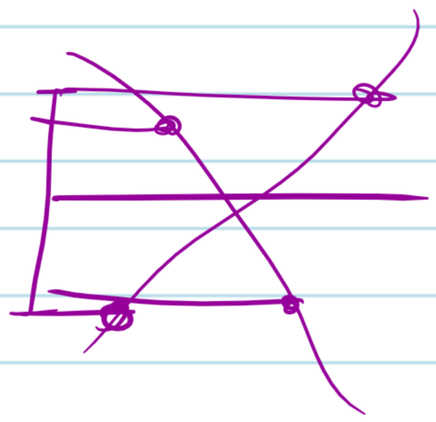
max ← $g(x_i-\varepsilon_i)$

$$= \int_{g(x_i-\varepsilon_i)} \delta(u)\,\frac{\phi(\bar g^{-1}(u))\,du}{g'(\bar g^{-1}(u))}$$

min

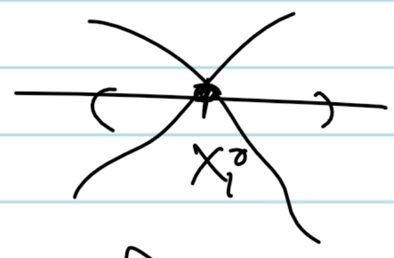
$g'(\bar g^{-1}(u))$ → +ve

$$\int_{U_i^0} \delta(g(x))\,\phi(x)\,dx = \int_{g(U_i^0)} \delta(u)\,\frac{\phi(\bar g^{-1}(u))}{|g'(\bar g^{-1}(u))|}\,du$$



$$= \left.\frac{\phi(\bar g^{-1}(u))}{|g'(\bar g^{-1}(u))|}\right|_{u=0} \quad \text{in } x \in U_i^0$$

$$\int_{U_i} \delta(g(x))\,\phi(x)\,dx = \left.\frac{\phi(\bar{g}^{-1}(u))}{|g'(\bar{g}^{-1}(u))|}\right|_{u=0}, \quad x \in U_i$$



$$= \frac{\phi(\bar{g}^{-1}(0))}{|g'(\bar{g}^{-1}(0))|}, \quad \boxed{x \in U_i}$$

$$\bar{g}^{-1}(0) = x_i^0$$

$$= \boxed{\frac{\phi(x_i^0)}{|g'(x_i^0)|}}$$

$\phi(a) = \langle \delta(x-a), \phi(x) \rangle$

Now, $\int_{\mathbb{R}} \delta(g(x))\,\phi(x)\,dx = \sum_i \left(\frac{\phi(x_i^0)}{|g'(x_i^0)|} \right)$

$$\langle \delta(g(x)), \phi(x) \rangle = \sum_i \frac{\langle \delta(x-x_i^0), \phi(x) \rangle}{|g'(x_i^0)|}$$

finite no. of real roots → I

non-repeated/simple roots → II

$g'(x_i^0)$ exists near x_i^0 → III

$$= \sum_i \left\langle \frac{\delta(x-x_i^0)}{|g'(x_i^0)|}, \phi(x) \right\rangle$$

$$= \left\langle \sum_i \frac{\delta(x-x_i^0)}{|g'(x_i^0)|}, \phi(x) \right\rangle$$

(due to II & III, $g'(x_i^0) \neq 0$.)

$$\boxed{\delta(g(x)) = \sum_i \frac{\delta(x-x_i^0)}{|g'(x_i^0)|}}, \text{ under I, II \& III.}$$

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|} \quad \text{where } g(x_i) = 0. \quad (x_i = \text{root})$$

→ using integration by substitution

This is valid when:

- (I) $g(x)$ has finitely many real roots, — nature of roots
- & (II) All roots of $g(x)$ are non-repeated (simple), — nature of roots
- & (III) $g'(x_i)$ exists (OR) $g'(x)$ exists near each x_i. — derivative near each root

→ necessary conditions

Integral form:

$$\int_{\mathbb{R}} \delta(g(x)) \cdot \phi(x)\,dx = \sum_i \frac{\phi(x_i)}{|g'(x_i)|}$$

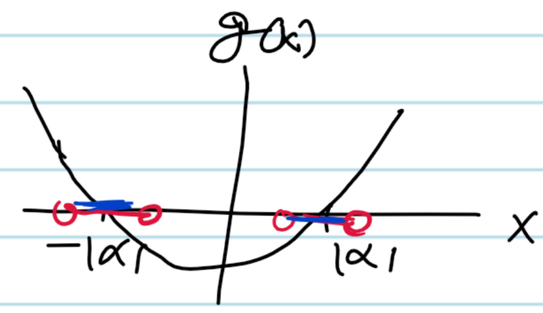
example $\delta(x^2 - \alpha^2) = ?$

$x, \alpha \in \mathbb{R}$ ✓

$x^2 - \alpha^2 = 0$

$x^2 = \alpha^2$

$x = |\alpha|, -|\alpha|$



[I] ✓

[II] ✓

[III] ✓

$g(x) = x^2 - \alpha^2$

$g'(x) = 2 \cdot x$

$g'(|\alpha|) = 2|\alpha| \xrightarrow{|\cdot|} 2 \cdot |\alpha|$

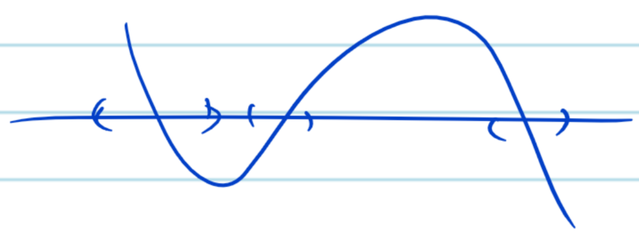
$g'(-|\alpha|) = -2|\alpha| \xrightarrow{|\cdot|} 2 \cdot |\alpha|$

$$\delta(x^2 - \alpha^2) = \frac{\delta(x - |\alpha|)}{2|\alpha|} + \frac{\delta(x + |\alpha|)}{2|\alpha|}$$

$$= \frac{1}{2|\alpha|} \cdot \left(\delta(x - |\alpha|) + \delta(x + |\alpha|)\right)$$

$$\boxed{\delta(x^2 - \alpha^2) = \frac{1}{2 \cdot |\alpha|} \cdot \left(\delta(x - \alpha) + \delta(x + \alpha)\right)} \qquad x, \alpha \in \mathbb{R}$$

Sufficient Conditions / Alternate Conditions



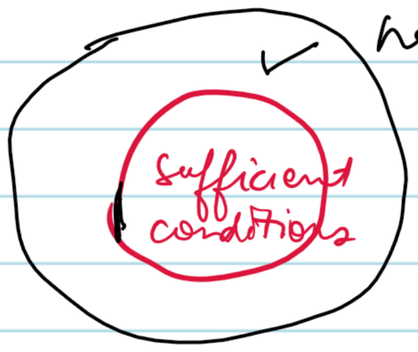
(I), (II) $\xrightarrow{\text{nature}}$ 1. $g(x)$ has finitely many real & simple roots.

as it is

differentiation of $g(x)$ $\xrightarrow{\quad}$ 2. $g(x)$ is continuously differentiable. $\begin{cases} g' \text{ exist} \\ g' \to \text{continuous} \end{cases}$

eg. $\delta(x^2 - \alpha^2)$, $x, \alpha \in \mathbb{R}$ } $|\alpha|, -|\alpha|$ ✓, $2x$ ✓ → formula ✓

[necessary ✓, alternate ✗] → formula ✓



Q.1 Show that $\int_{\mathbb{R}} \delta(g(x)) \cdot |g'(x)| \cdot \varphi(g(x))\,dx = \varphi(0) \times N_g$, where N_g is no. of zeros of $g(x)$; under the assumption that $g(x): \mathbb{R} \to \mathbb{R}$ is continuously differentiable (III), N_g is finite (I), and $g'(x)$ is non-zero near the roots (II).

$\phi(x)$

using formula $= \sum_i \frac{\phi(x_i)}{|g'(x_i)|} = \sum_i \frac{|g'(x_i)| \cdot \varphi(g(x_i))}{|g'(x_i)|}$

$= \sum_i \varphi(g(x_i)) = \sum_i \varphi(0)$ ($g(x_i) \to 0$)

$= \varphi(0) \cdot \left(\sum_i 1\right)$

$= N_g \cdot \varphi(0)$

Q.2 (i) Show that $\int_{\mathbb{R}} \delta(g(x)) \cdot |g'(x)| \cdot \varphi(x)\,dx = \sum_i \varphi(x_i)$ where x_i is a zero of $g(x) = 0$.

Assume $g(x) \in C^1$, $g'(x_i) \neq 0 \ \forall x_i$.

(ii) What if the assumptions were: $g(x) \in C^1$, $g'(x) \neq 0 \ \forall x \in \mathbb{R}$

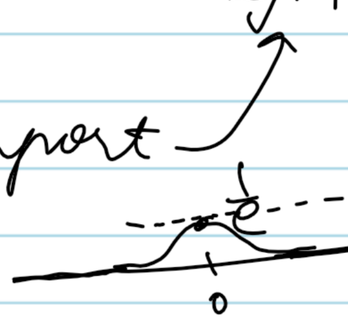
---

notations: $C^1 \to g'(x)$ exists & $g'(x)$ is continuous.

$C^\infty \to$ smooth (eg. sinusoids)

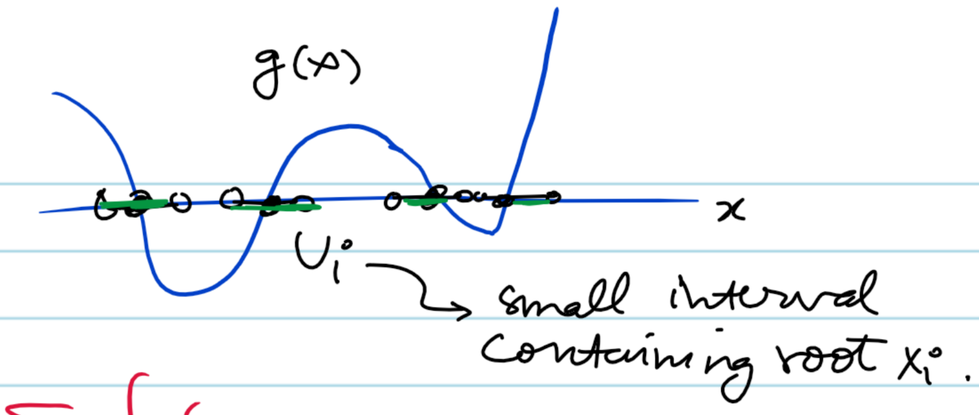
$C_c^\infty \to$ smooth with compact support (eg. test functions.)

$$\begin{cases} e^{\frac{-1}{1-x^2}}, & |x| < 1 \\ 0, & \text{else.} \end{cases}$$



$\Rightarrow C^k \to g^{(k)}(x)$ exists & is continuous.

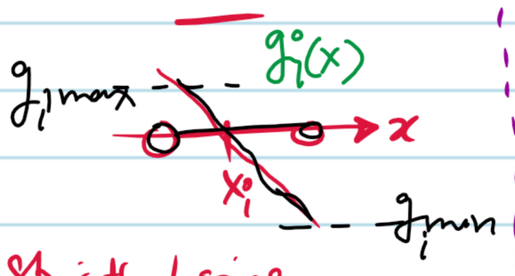
Q.1 : method 2 (no-formula used)



$$\int_{\mathbb{R}} \delta(g(x)) \cdot |g'(x)| \, \varphi(g(x))\,dx = \begin{cases} \sum_i \int_{U_i^\circ} \delta(g(x)) \cdot |g'(x)| \, \varphi(g(x))\,dx \\ 0, \text{ if } g(x) \text{ has no real roots.} \end{cases}$$

→ Within U_i°, due to assumptions, $g(x)$ is strictly monotonous.

→ c-i

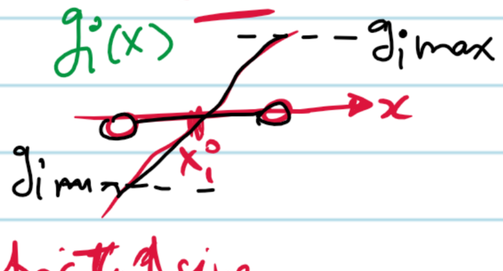


strictly ↓ sing

$g'(x) < 0$ $x \in U_i$

$$= -\int_{g_{i\max}}^{g_{i\min}} \delta(u)\,\varphi(u)\,du = \int_{g(U_i^\circ)} \delta(u)\,\varphi(u)\,du$$

c-ii



strictly ↑ sing

$g'(x) > 0$ $x \in U_i$

$$= \int_{g_{i\min}}^{g_{i\max}} \delta(u)\,\varphi(u)\,du = \int_{g(U_i^\circ)} \delta(u)\,\varphi(u)\,du$$

integration by substitution $g(x) = u$.

$$\text{in: } \int_{U_i^\circ} \delta(g(x))\,|g'(x)|\,\varphi(g(x))\,dx = \int_{g_i(x_i-\varepsilon_i')}^{g_i(x_i+\varepsilon_i)} \delta(u)\,\varphi(u)\,\frac{|g'(x)|}{g'(x)}\,du$$

sign of $g'(x)$.

$$= \int_{-\nu}^{+\nu} \delta(u)\,\varphi(u)\,du$$

$$\Rightarrow \int_{U_i^\circ} \delta(g(x))\,|g'(x)|\,\varphi(g(x))\,dx = \int_{g(U_i^\circ)} \delta(u)\,\varphi(u)\,du = \langle \delta(u), \varphi(u) \rangle = \varphi(0).$$

thus, $$\int_{\mathbb{R}} \delta(g(x)) \cdot |g'(x)|\,\varphi(g(x))\,dx = \varphi(0) \times \sum_i \frac{1}{|g'(x_i)|} = \varphi(0) \times N_g$$

Q.2 (i) Show that $\int_{\mathbb{R}} \delta(g(x)) \cdot |g'(x)| \cdot \varphi(x)\,dx = \sum_i \varphi(x_i)$ where x_i is a zero of $g(x)=0$.

$|g'(x)| \cdot \varphi(x) \to \phi(x)$

Assume $g(x) \in C^1$ (III), $g'(x_i) \neq 0 \ \forall x_i$ (II).

(ii) What if the assumptions were: $g(x) \in C^1$, $g'(x) \neq 0 \ \forall x \in \mathbb{R}$, and [g is an algebraic polynomial function.]

2-i formula, $\sum_i \frac{\phi(x_i^o)}{|g'(x_i^o)|} = \sum_i \varphi(x_i^o)$ ✓

2-ii $g(x) \in C^1$

$g'(x) \neq 0$

$g(x)$ has a single real root $\bar{g}^1(0)$ ⟸ strictly monotonic $\forall x$

$$\int_{\mathbb{R}} \delta(g(x)) \, |g'(x)| \, \varphi(x)\,dx = \varphi(\bar{g}^1(0)) \ \checkmark$$

2+(ii)

---

property - 7

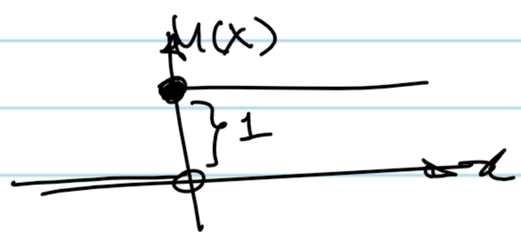
Indefinite integral [edit]

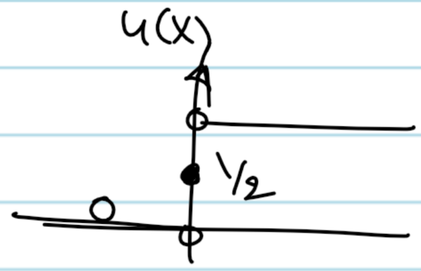
For a constant $a \in \mathbb{R}$ and a "well-behaved" arbitrary real-valued function $y(x)$,

$$\int y(x)\delta(x-a)dx = y(a)H(x-a) + c,$$

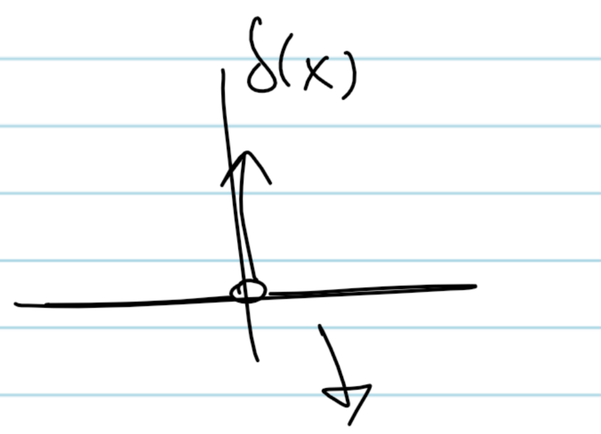
where $H(x)$ is the Heaviside step function and c is an integration constant.

$$u(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$





$$h(x) = \frac{dH}{dx} = \begin{cases} 0, & x > 0 \\ 0, & x < 0 \\ \text{DNE}, & x = 0 \end{cases} =$$

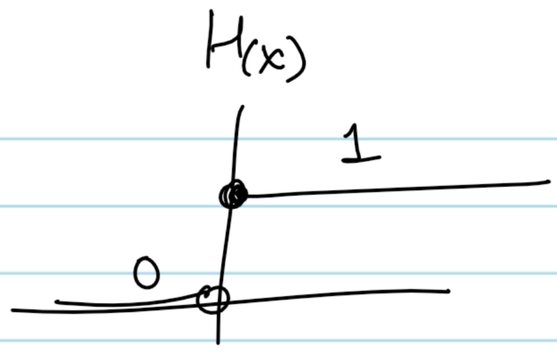


$$\int_{-\infty}^{\infty} h(x)\,dx = H(x)\Big|_{-\infty}^{\infty} = H(\infty) - H(-\infty)$$

$H(-\infty) \to 0$

area = 1

$$\boxed{\delta(x) = \frac{dH}{dx}}$$



$$\delta(x) = \frac{dH}{dx}$$

$$\int \delta(x)\,dx = H(x) + C$$

$$H(x) = \begin{cases} \int_{-\infty}^{x} \delta(t)\,dt & , x \neq 0 \\ 1 \text{ or } \frac{1}{2} & , x = 0 \end{cases}$$

example -1 $\int \underbrace{\delta(x) \cdot y(x)}\,dx = ?$

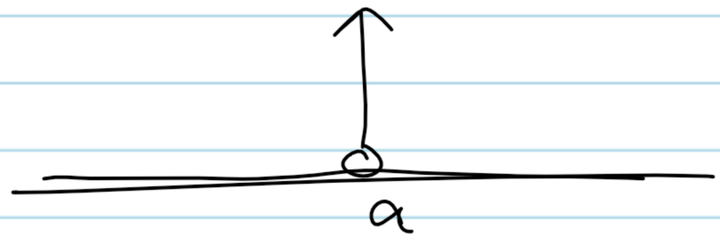
$$y(0) \times \delta(x) = \int y(0) \times \delta(x)\,dx$$

$$= y(0) \cdot \underbrace{\int \delta(x)\,dx} = y(0) \times H(x) + C$$

$$\therefore \boxed{\int \delta(x) \cdot y(x)\,dx = y(0) \times H(x) + C}$$

Example -2 $\int \underbrace{\delta(x-a) \cdot y(x)}\,dx = ?$

$\rightarrow y(a) \times \delta(x-a)$



area $= y(a)$.

$$= \int y(a) \times \delta(x-a)\,dx = y(a) \times \underbrace{\int \delta(x-a)\,dx}$$

$x - a = t$

$$\int \delta(t)\,dt = H(t) + C$$

$$= H(x-a) + C.$$

$$\therefore \boxed{\int \delta(x-a)\,y(x)\,dx = y(a) \times H(x-a) + C}$$